

Elliptic Curves

(PARI-GP version 2.10.0)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E = ellinit**($v, \{D = 1\}$)
over **Q** $D = 1$
over **F_p** $D = p$
over **F_q**, $q = p^f$ $D = \text{ffgen}([p, f])$
over **Q_p**, precision n $D = O(p^n)$
over **C**, current bitprecision $D = 1.0$
over number field K $D = nf$

Points are [x,y], the origin is [0]. Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- E defined over **R** or **C**
 x -coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega, E.eta**
volume of complex lattice **E.area**

- E defined over **Q_p**
residual characteristic **E.p**
If $|j_p| > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**

- E defined over **F_q**
characteristic **E.p**
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**

- E defined over **Q**
generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant **ellfromj**(j)
cubic/quartic/biquadratic to Weierstrass **ellfromeqn**(eq)
add points $P + Q$ / $P - Q$ **elladd**(E, P, Q), **ellsub**
negate point **ellneg**(E, P)
compute $n \cdot z$ **ellmul**(E, z, n)
check if z is on E **ellisoncurve**(E, z)
order of torsion point z **ellorder**(E, z)
 y -coordinates of point(s) for x **ellordinate**(E, x)
point $[\wp(z), \wp'(z)]$ corresp. to z **ellztopoint**(E, z)
complex z such that $p = [\wp(z), \wp'(z)]$ **ellpointtoz**(E, p)

- **Change of Weierstrass models, using** $v = [u, r, s, t]$
change curve E using v **ellchangecurve**(E, v)
change point z using v **ellchangept**(z, v)
change point z using inverse of v **ellchangeptinv**(z, v)

- **Twists and isogenies**
quadratic twist **elltwtst**(E, D)
 n -division polynomial $f_n(x)$ **elldivpol**($E, n, \{x\}$)
 $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$; return (ϕ_n, ψ_n^2) **ellxn**(E, n, v)
isogeny from E to E/G **ellisogeny**(E, G)
apply isogeny to g (point or isogeny) **ellisogenyapply**(f, g)

- **Formal group**
formal exponential, n terms **ellformalexp**($E, \{n\}, \{v\}$)
formal logarithm, n terms **ellformalog**($E, \{n\}, \{v\}$)
 $L(-x/y) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog**(E, p, n, P)
 $[x, y]$ in the formal group **ellformalpoint**($E, \{n\}, \{v\}$)
 $[f, g], \omega = f(t)dt, x\omega = g(t)dt$ **ellformaldifferential**
 $w = -1/y$ in parameter $-x/y$ **ellformalw**($E, \{n\}, \{v\}$)

Curves over finite fields, Pairings

random point on E **random**(E)
 $\#E(\mathbf{F}_q)$ **ellcard**(E)
 $\#E(\mathbf{F}_q)$ with almost prime order **ellsea**($E, \{\text{tors}\}$)
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup**(E)
is E supersingular? **ellissupersingular**(E)
Weil pairing of m -torsion pts x, y **ellweilpairing**(E, x, y, m)
Tate pairing of x, y ; x m -torsion **elltatepairing**(E, x, y, m)
Discrete log, find n s.t. $P = [n]Q$ **elllog**($E, P, Q, \{\text{ord}\}$)

Curves over Q

Reduction, minimal model
minimal model of E/\mathbf{Q} **ellminimalmodel**($E, \{\&v\}$)
quadratic twist of minimal conductor **ellminimaltwist**
multiple with good reduction **ellnonsingularmultiple**(E, P)

Complex heights
canonical height of P **ellheight**(E, P)
canonical bilinear form taken at P, Q **ellheight**(E, P, Q)
height regulator matrix for pts in x **ellheightmatrix**(E, x)

p -adic heights
cyclotomic p -adic height of $P \in E(\mathbf{Q})$ **ellpadicheight**(E, P, n)
... bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight**(E, P, n, Q)
... matrix at vector of points **ellpadicheightmatrix**(E, p, n, x)
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobenius**(E, p, n)
slope of unit eigenvector of Frobenius **ellpadics2**(E, p, n)

Isogenous curves
matrix of isogeny degrees for \mathbf{Q} -isog. curves **ellisomat**(E)
a modular equation of prime degree N **ellmodulareqn**(N)

L -function
 p -th coeff a_p of L -function, p prime **ellap**(E, p)
 E supersingular at p ? **ellissupersingular**(E, p)
 k -th coeff a_k of L -function **ellak**(E, k)
 $L(E, s)$ (using less memory than **lfun**) **elllseries**(E, s)
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **elll1**(E, r)
a Heegner point on E of rank 1 **ellheegner**(E)
order of vanishing at 1 **ellanalyticrank**($E, \{\text{eps}\}$)
root number for $L(E, \cdot)$ at p **ellrootno**($E, \{p\}$)
modular parametrization of E **elltaniyama**(E)
degree of modular parametrization **ellmoddegree**(E)
 p -adic L -function of E at χ^s **ellpadicL**($E, p, n, \{s = 0\}$)

Elldata package, Cremona's database:
db code "11a1" \leftrightarrow [conductor, class, index] **ellconvertname**(s)
generators of Mordell-Weil group **ellgenerators**(E)
look up E in database **ellidentify**(E)
all curves matching criterion **ellsearch**(N)
loop over curves with cond. from a to b **forell**(E, a, b, seq)

Curves over number field K

coeff a_p of L -function **ellap**(E, p)
Kodaira type of \mathfrak{p} -fiber of E **elllocalred**(E, p)
integral model of E/K **ellintegralmodel**($E, \{\&v\}$)
minimal model of E/K **ellminimalmodel**($E, \{\&v\}$)
cond, min mod, Tamagawa num $[N, v, c]$ **ellglobalred**(E)
 $P \in E(K)$ n -divisible? $[n]Q = P$ **ellisdivisible**($E, P, n, \{\&Q\}$)

L-function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h$; $D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
vector of first n a_k 's in L -function **elllan**(E, n)
init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfuninit**($E, D, \{n = 0\}$)
compute $L(E, s)$ (n -th derivative) **lfun**($L, s, \{n = 0\}$)
torsion subgroup with generators **elltors**(E)

Other curves of small genus

A hyperelliptic curve is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
reduction of $y^2 + Qy = P$ (genus 2) **genus2red**($[P, Q], \{p\}$)
find a rational point on a conic, ${}^t_x Gx = 0$ **qfsolve**(G)
quadratic Hilbert symbol (at p) **hilbert**($x, y, \{p\}$)
all solutions in \mathbf{Q}^3 of ternary form **qfparam**(G, x)
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius **hyperellcharpoly**($[P, Q]$)
matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ **hyperellpadicfrobenius**

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean **agm**(x, y)
elliptic j -function $1/q + 744 + \dots$ **ellj**(x)
Weierstrass $\sigma/\wp/\zeta$ function **ellsigma**(w, z), **ellwp**, **ellzeta**
periods/quasi-periods **ellperiods**($E, \{\text{flag}\}$), **ellleta**(w)
 $(2i\pi/\omega_2)^k E_k(\tau)$ **elleisnum**($w, k, \{\text{flag}\}$)
modified Dedekind η func. $\prod(1 - q^n)$ **eta**($x, \{\text{flag}\}$)
Dedekind sum $s(h, k)$ **sumdedekind**(h, k)
Jacobi sine theta function **theta**(q, z)
 k -th derivative at $z=0$ of $\theta(q, z)$ **thetanullk**(q, k)
Weber's f functions **weber**($x, \{\text{flag}\}$)
modular pol. of level N **polmodular**($N, \{\text{inv} = j\}$)
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ **polclass**($D, \{\text{inv} = j\}$)

Based on an earlier version by Joseph H. Silverman
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